Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Computational and Applied Mathematics 192 (2006) 30–39
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Numerical analysis and simulations of a quasistatic frictional contact problem with damage in viscoelasticity

M. Campo*, J.R. Fernández, J.M. Viaño

Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, Facultade de Matemáticas, Campus Sur s/n, 15782 Santiago de Compostela, Spain

Received 15 September 2004; received in revised form 1 March 2005

Abstract

We consider a quasistatic problem which models the bilateral contact between a viscoelastic body and a foundation, taking into account the damage and the friction. The damage which results from tension or compression is then involved in the constitutive law and it is modelled using a nonlinear parabolic inclusion. The variational problem is formulated as a coupled system of evolutionary equations for which we state the existence of a unique solution. Then, we introduce a fully discrete scheme using the finite element method to approximate the spatial variable and the Euler scheme to discretize the time derivatives. Error estimates are derived and, under suitable regularity hypotheses, the convergence of the numerical scheme obtained. Finally, a numerical algorithm and results are presented for some two-dimensional examples.

Keywords: Quasistatic bilateral contact; Damage; Friction Tresca's law; Viscoelastic; Error estimates; Numerical simulations

1. Introduction

In this work we study, from both variational and numerical point of views, a model for the process of bilateral contact between a viscoelastic body and a foundation, when the material damage due to compression or tension and the friction are taken into account. Reliable prediction of the development of material damage resulting from the opening and growth of microcracks is of considerable interest to the engineering (see [8,12–14] and the monograph [7]). The effective functioning, reliability and safety

* Corresponding author.
E-mail address: macampo@usc.es (M. Campo).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2005.04.046

of the system may be affected by the decrease in its load carrying capability, as microcracks grow when the material undergoes damaged.

We assume that the contact with the obstacle, the so-called foundation, is always produced and then, a bilateral contact condition is considered. Moreover, the friction is modelled using classical Tresca's law with a given friction bound. This condition has been considered in other physical settings (see, e.g., [1,6]). Finally, the inertia effects are assumed to be negligible and so the process is quasistatic.

One of the main objectives of this paper is to develop an efficient algorithm for solving this frictional contact problem. In [2] an augmented lagrangean method was used. Here, we apply a penalty method coupled with the penalty-duality algorithm introduced in [3].

2. Physical setting and variational formulation

We denote by S_d the space of second-order symmetric tensors on $\mathbb{R}^d$. Let "$\cdot$" be the inner product on $\mathbb{R}^d$ or S_d, and $|\cdot|$ the Euclidean norms on these spaces.

We consider a viscoelastic body that occupies the domain $\Omega \subset \mathbb{R}^d$, and let the time interval of interest be $[0, T]$, $T > 0$. The outer surface $\Gamma = \partial\Omega$ is assumed to be Lipschitz continuous, and it is divided into three disjoint measurable parts Γ_D, Γ_N and Γ_C, where $\text{meas}(\Gamma_D) \neq \emptyset$. For a.e. $\boldsymbol{x} \in \Gamma$, we denote by $\boldsymbol{\nu}(\boldsymbol{x})$ and $\boldsymbol{\tau}(\boldsymbol{x})$ the unit outward normal and tangential vectors to Γ, respectively. The body is clamped on Γ_D, and so the displacement field vanishes there, surface tractions of density $\boldsymbol{f}_N$ act on Γ_N, and a volume forces density $\boldsymbol{f}_B$ acts in $\Omega_T = \Omega \times (0, T)$. Finally, the body is assumed to be in bilateral contact with a foundation over the contact surface Γ_C (see Fig. 1).

We denote by $\boldsymbol{u}$ the displacement field, $\boldsymbol{\sigma}$ the stress tensor and $\boldsymbol{\epsilon}(\boldsymbol{u})$ the linearized strain tensor. Moreover, let ζ be the damage field, defined in Ω, that measures the density of the microcracks in the material, and will be described below. The material is assumed viscoelastic with the following constitutive law ([6,11]):

$$\boldsymbol{\sigma} = \mathscr{A}(\boldsymbol{\epsilon}(\dot{\boldsymbol{u}})) + \mathscr{G}(\boldsymbol{\epsilon}(\boldsymbol{u}), \zeta),$$

where $\mathscr{A}$ and $\mathscr{G}$ are the viscosity and elasticity constitutive functions, respectively, which will be described below. Here, a dot above a variable represent its partial time derivative.

The damage of the material measures the density of the microcracks: when $\zeta = 1$ the material is in its undamaged state, when $\zeta = 0$ the material is fully damaged and when $0 < \zeta < 1$ there is partial damage. According to [8], the evolution of the damage is governed by the following parabolic nonlinear differential

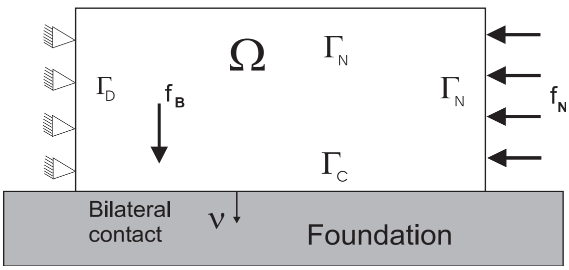

Fig. 1. Physical setting of a viscoelastic body in frictional contact with damage.

inclusion:

$$\dot{\zeta} - \kappa\Delta\zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\epsilon}(\boldsymbol{u}), \zeta).$$

Here, Δ is the Laplacian, $\kappa > 0$ is the damage diffusion constant, ϕ is the damage source function and $\partial I_{[0,1]}$ denotes the subdifferential of the indicator function $I_{[0,1]}$ of the interval [0, 1].

We assume that there is no damage influx throughout the boundary Γ, and therefore $\partial\zeta/\partial\boldsymbol{v} = 0$ on Γ. Also, we postulate a lower limit for the damage, β_*, as when that value is reached, modelling the material as viscoelastic becomes inadequate.

Let $\boldsymbol{u}_0$ and ζ_0 be the initial values of the displacement and damage fields, respectively, and assume that the inertia effects are negligible and so the process is quasistatic. Thus, the classical form of the mechanical problem of quasistatic frictional bilateral contact with damage of a viscoelastic body with a foundation is as follows.

Problem P. Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to S_d$, and a damage field $\zeta : \Omega \times [0, T] \to [0, 1]$ such that,

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_B = \boldsymbol{0} \quad \text{in } \Omega_T, \tag{1}$$

$$\boldsymbol{\sigma} = \mathscr{A}(\boldsymbol{\epsilon}(\dot{\boldsymbol{u}})) + \mathscr{G}(\boldsymbol{\epsilon}(\boldsymbol{u}), \zeta) \quad \text{in } \Omega_T, \tag{2}$$

$$\dot{\zeta} - \kappa\Delta\zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\epsilon}(\boldsymbol{u}), \zeta) \quad \text{in } \Omega_T, \tag{3}$$

$$\frac{\partial\zeta}{\partial\boldsymbol{v}} = 0 \quad \text{on } \Gamma \times (0, T), \tag{4}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_D \times (0, T), \tag{5}$$

$$\boldsymbol{\sigma v} = \boldsymbol{f}_N \quad \text{on } \Gamma_N \times (0, T), \tag{6}$$

$$\left.\begin{array}{l} u_v = 0, \quad |\boldsymbol{\sigma}_\tau| \leqslant g, \\ |\boldsymbol{\sigma}_\tau| < g \Rightarrow \dot{\boldsymbol{u}}_\tau = \boldsymbol{0}, \\ |\boldsymbol{\sigma}_\tau| = g \Rightarrow \text{there exists } \lambda > 0; \quad \boldsymbol{\sigma}_\tau = -\lambda \dot{\boldsymbol{u}}_\tau \end{array}\right\} \quad \text{on } \Gamma_C \times (0, T), \tag{7}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \zeta(0) = \zeta_0 \quad \text{in } \Omega. \tag{8}$$

Here, Eq. (7) represent the classical Tresca's condition, where $u_v = \boldsymbol{u} \cdot \boldsymbol{v}$ denotes the normal displacement, $\boldsymbol{\sigma}_\tau = (\boldsymbol{\tau} \cdot \boldsymbol{\sigma v})\boldsymbol{\tau}$ and $\dot{\boldsymbol{u}}_\tau = (\boldsymbol{\tau} \cdot \dot{\boldsymbol{u}})\boldsymbol{\tau}$ are the tangential components of the stress and velocity fields, respectively, and g represents a friction bound.

Now, we introduce additional notation and the assumptions on the problem data. Let us define the variational spaces:

$$H = [L^2(\Omega)]^d, \quad Y = L^2(\Omega), \quad \mathscr{K} = \{\xi \in H^1(\Omega);\ 0 \leqslant \xi \leqslant 1 \text{ a.e. in } \Omega\},$$

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^2;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D,\ v_v = \boldsymbol{v} \cdot \boldsymbol{v} = 0 \text{ on } \Gamma_C\},$$

$$Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^2 \in [L^2(\Omega)]^{2\times 2};\ \tau_{ij} = \tau_{ji},\ i, j = 1, 2\}.$$

Moreover, for a Banach space X, let $(\cdot, \cdot)_X$ denote its inner product and $\|\cdot\|_X$ its associated norm.

In the study of the mechanical problem (1)–(8), we assume essentially that the viscosity operator $\mathscr{A}$, the elasticity operator $\mathscr{G}$ and the damage source function ϕ are Lipschitz continuous operators and that $\mathscr{A}$ is strictly monotone.

Let the body forces and surface tractions have the regularity $\boldsymbol{f}_B \in \mathscr{C}([0,T];H)$ and $\boldsymbol{f}_N \in \mathscr{C}([0,T];[L^2(\Gamma_N)]^d)$, define the element $\boldsymbol{f}(t) \in V$ given by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{f}_B(t), \boldsymbol{v})_H + (\boldsymbol{f}_N(t), \boldsymbol{v})_{[L^2(\Gamma_N)]^d},$$

and let $g : \Gamma_C \to [0, +\infty)$ be such that $g \in L^\infty(\Gamma_C)$.

Let $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be the bilinear form

$$a(\zeta_1, \zeta_2) = \kappa \int_\Omega \nabla\zeta_1 \cdot \nabla\zeta_2 \,\mathrm{d}x, \quad \forall \zeta_1, \zeta_2 \in H^1(\Omega),$$

and we denote by $j : V \to \mathbb{R}$ the functional

$$j(\boldsymbol{v}) = \int_{\Gamma_C} g|v_\tau| \,\mathrm{d}S \quad \forall \boldsymbol{v} \in V.$$

Choosing test functions from V and $\mathscr{K}$, applying the Green's formula and using the conditions and the notation above, we obtain the following formulation.

Problem VP. Find a displacement field $\boldsymbol{u} : [0,T] \to V$, a stress field $\boldsymbol{\sigma} : [0,T] \to Q$, and a damage field $\zeta : [0,T] \to \mathscr{K}$, such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\zeta(0) = \zeta_0$ and for a.e. $t \in [0,T]$,

$$\begin{aligned}
&\boldsymbol{\sigma}(t) = \mathscr{A}(\epsilon(\dot{\boldsymbol{u}}(t))) + \mathscr{G}(\epsilon(\boldsymbol{u}(t)), \zeta(t)),\\
&(\boldsymbol{\sigma}(t), \epsilon(\boldsymbol{w} - \dot{\boldsymbol{u}}(t)))_Q + j(\boldsymbol{w}) - j(\dot{\boldsymbol{u}}(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_V, \quad \forall \boldsymbol{w} \in V,\\
&(\dot{\zeta}(t), \xi - \zeta(t))_Y + a(\zeta(t), \xi - \zeta(t)) \geqslant (\phi(\epsilon(\boldsymbol{u}(t)), \zeta(t)), \xi - \zeta(t))_Y, \quad \forall \xi \in \mathscr{K}.
\end{aligned}$$

The existence of a unique solution to Problem **VP** and its regularity are summarized in the following theorem.

Theorem 1. *If the initial conditions are chosen in such a way that $\boldsymbol{u}_0 \in V$ and $\zeta_0 \in \mathscr{K}$, then Problem* **VP** *has a unique solution in such a way that $\boldsymbol{u} \in C^1([0,T];V)$ and $\zeta \in H^1(0,T;Y) \cap L^2(0,T;H^1(\Omega))$.*

The proof of Theorem 1 is obtained following [10] and it is based on monotone operator theory, classical results on parabolic equations and Banach fixed point arguments.

3. Numerical approximations

In this section we introduce a finite element algorithm for solving Problem **VP** and obtain an error estimate on the approximate solutions. For convenience, we rewrite the variational problem **VP** in terms of the velocity field $\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t)$ given by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\,\mathrm{d}s + \boldsymbol{u}_0.$$

The discretization of this variational problem will be done in two steps. First, we consider two finite-dimensional spaces $V^h \subset V$ and $B^h \subset H^1(\Omega)$, approximating the spaces V and $H^1(\Omega)$, respectively. Let $\mathscr{K}^h = \mathscr{K} \cap B^h$. Here, $h > 0$ denotes the discretization parameter.

To discretize the time derivatives, we consider a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$ and let k be the time step size, $k = T/N$. For a continuous function $f(t)$, let $f_n = f(t_n)$ and, for a sequence $\{w_n\}_{n=0}^N$, we let $\delta w_n = (w_n - w_{n-1})/k$ be its corresponding divided differences.

The fully discrete approximation of Problem **VP**, based on the forward Euler scheme, is as follows.

Problem VP^{hk}. Find $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ and $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset \mathscr{K}^h$, such that $\zeta_0^{hk} = \zeta_0^h$ and for all $\xi^h \in \mathscr{K}^h$, $\boldsymbol{w}^h \in V^h$ and $n = 1, 2, \ldots, N$,

$$(\delta\zeta_n^{hk}, \xi^h - \zeta_n^{hk})_Y + a(\zeta_n^{hk}, \xi^h - \zeta_n^{hk}) \geqslant (\phi(\boldsymbol{\epsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_Y,$$
$$(\mathscr{A}(\boldsymbol{\epsilon}(\boldsymbol{v}_n^{hk})) + \mathscr{G}(\boldsymbol{\epsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\epsilon}(\boldsymbol{w}^h - \boldsymbol{v}_n^{hk}))_Q + j(\boldsymbol{w}^h) - j(\boldsymbol{v}_n^{hk}) \geqslant (\boldsymbol{f}_n, \boldsymbol{w}^h - \boldsymbol{v}_n^{hk})_V,$$

where the discrete displacement fields $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ are defined by $\boldsymbol{u}_n^{hk} = \boldsymbol{u}_{n-1}^{hk} + k\boldsymbol{v}_n^{hk}$ for $n = 1, 2, \ldots, N$, and $\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h$ and ζ_0^h are appropriate approximations of the initial conditions.

Using standard arguments for variational inequalities (see [9]), we deduce the existence and uniqueness of the solution to Problem **VP^{hk}**.

Applying similar arguments to those employed in [4,10], after some algebra it leads to the following.

Theorem 2. *Let the hypotheses of Theorem* 1 *hold. Assume that*

$$\zeta \in C([0, T]; H^2(\Omega)) \cap H^2(0, T; Y).$$

Then, the following error estimate is obtained for all $\{\xi_j^h\}_{j=0}^N \subset \mathscr{K}^h$ *and* $\{\boldsymbol{w}_j^h\}_{j=0}^N \subset V^h$,

$$\begin{aligned}
&\max_{0\leqslant n\leqslant N}\{\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2\} + k\sum_{j=1}^N \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 \\
&\leqslant c\left\{\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\zeta_0 - \zeta_0^h\|_Y^2 + \|\zeta_1 - \xi_1^h\|_Y^2 + \max_{0\leqslant n\leqslant N}\|\zeta_n - \xi_n^h\|_Y^2\right. \\
&\quad + \max_{1\leqslant n\leqslant N} I_n^2 + k\sum_{j=1}^N \|\zeta_j - \xi_j^h\|_{H^1(\Omega)}^2 + k^2(\|\zeta\|_{H^2(0,T;Y)}^2 + \|\dot{\boldsymbol{u}}\|_{C([0,T];V)}^2) \\
&\quad + \frac{1}{k}\sum_{j=1}^{N-1}\|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 + \max_{0\leqslant n\leqslant N}\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V \\
&\quad \left. + k\sum_{j=1}^N \left\|\phi(\boldsymbol{\epsilon}(\boldsymbol{u}_j), \zeta_j) - \frac{\zeta_j - \zeta_{j-1}}{k} + \kappa\Delta\zeta_j\right\|_Y \cdot \|\zeta_j - \xi_j^h\|_Y\right\},
\end{aligned}$$

where

$$I_n = \| \int_0^{t_{n-1}} \boldsymbol{v}(s)\,\mathrm{d}s - \sum_{j=1}^{n-1} k\boldsymbol{v}_j \|_V .$$

The above error estimates is the basis for the convergence analysis. As an example, assume that Ω is a polygonal domain, $\mathscr{T}^h$ denotes a finite element triangulation of $\overline{\Omega}$, and V^h and B^h are finite element spaces made of continuous and piecewise affine functions. Therefore, we have the following error estimate result.

Corollary 3. *Under the assumptions of Theorem* 2, *let the discrete initial conditions be chosen as* $\boldsymbol{u}_0^h = \Pi^h \boldsymbol{u}_0$, $\zeta_0^h = \pi^h \zeta_0$, *where* $\pi^h : C(\overline{\Omega}) \to B^h$ *is the standard finite element interpolation operator* (*see* [5]) *and* $\Pi^h = (\pi_i^h)_{i=1}^d : [C(\overline{\Omega})]^d \to V^h$. *If we assume that*

$$\boldsymbol{u} \in H^2(0,T;V) \cap C^1([0,T];[H^2(\Omega)]^d),$$
$$\zeta \in C([0,T];H^2(\Omega) \cap H^2(0,T;Y), \quad \dot{\zeta} \in L^2(0,T;H^1(\Omega)),$$

we have the following error estimate,

$$\max_{0 \leqslant n \leqslant N} \{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\zeta_n - \zeta_n^{hk}\|_Y \} \leqslant c(h^{1/2} + k).$$

Moreover, *if* $\boldsymbol{\sigma\nu} \in C([0,T];[L^2(\Gamma)]^d)$ *and* $\dot{\boldsymbol{u}}_\tau \in C([0,T];[H^2(\Gamma_C)]^d)$, *then*

$$\max_{0 \leqslant n \leqslant N} \{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\zeta_n - \zeta_n^{hk}\|_Y \} \leqslant c(h + k).$$

4. Numerical resolution of Problem VP^{hk} ($d = 2$)

Let $n \in \{1, \dots, N\}$ and assume that $\boldsymbol{u}_{n-1}^{hk}$ and ζ_{n-1}^{hk} are known. First, from Problem **VP**^{hk} we obtain that the discrete damage field is the unique solution of the following problem,

$$(\zeta_n^{hk}, \xi^h - \zeta_n^{hk})_Y + ka(\zeta_n^{hk}, \xi^h - \zeta_n^{hk}) \geqslant k(\phi(\boldsymbol{\epsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_Y + (\zeta_{n-1}^{hk}, \xi^h - \zeta_n^{hk})_Y, \quad \forall \xi^h \in \mathscr{K}^h. \tag{9}$$

Problem (9) is a classical first-kind variational inequality which has been solved using a penalty-duality algorithm introduced in [3].

Secondly, the discrete velocity field is obtained solving the following variational inequality,

$$(\mathscr{A}(\boldsymbol{\epsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\epsilon}(\boldsymbol{w}^h - \boldsymbol{v}_n^{hk}))_Q + j(\boldsymbol{w}^h) - j(\boldsymbol{v}_n^{hk}) \geqslant (\boldsymbol{f}_n, \boldsymbol{w}^h - \boldsymbol{v}_n^{hk})_V - (\mathscr{G}(\boldsymbol{\epsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\epsilon}(\boldsymbol{w}^h - \boldsymbol{v}_n^{hk}))_Q, \quad \forall \boldsymbol{w}^h \in V^h. \tag{10}$$

We note that problem (10) is a second-kind variational inequality. For numerical algorithms to solve it we refer to [9] and bibliography therein. Nevertheless, here we introduce an efficient penalty method coupled with a penalty-duality algorithm that is only valid for two-dimensional problems ($d = 2$). The main idea is to define a penalized friction condition as $-\sigma_\tau = \Phi_\mu(\dot{u}_\tau)$ where, for $0 < \mu$, $\Phi_\mu(r) = -g$ if

$r < -\mu$, $\Phi_\mu(r) = \frac{g}{\mu} r$ if $r \in [-\mu, \mu]$ and $\Phi_\mu(r) = g$ if $r > \mu$. Moreover, we denote by $\sigma_\tau = \boldsymbol{\sigma}_\tau \cdot \boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{v} \cdot \boldsymbol{\tau}$ and $\dot{u}_\tau = \dot{\boldsymbol{u}}_\tau \cdot \boldsymbol{\tau} = \dot{\boldsymbol{u}} \cdot \boldsymbol{\tau}$ the respective tangential projections of the shear stresses and velocity field.

$$\text{For all } \boldsymbol{w} \in V, \text{ let } w_\tau = \boldsymbol{w} \cdot \boldsymbol{\tau} \text{ and } \boldsymbol{u}_\mu(t) = \int_0^t \boldsymbol{v}_\mu(s)\,\mathrm{d}s + \boldsymbol{u}_0.$$

Using the above condition, we obtain the following nonlinear variational equation for the velocity field,

$$\begin{aligned}
&(\mathscr{A}(\boldsymbol{\epsilon}(\boldsymbol{v}_\mu(t))), \boldsymbol{\epsilon}(\boldsymbol{w}))_Q + \int_{\Gamma_C} \Phi_\mu(\boldsymbol{v}_\mu \cdot \boldsymbol{\tau}) w_\tau \,\mathrm{d}a \\
&\quad = (\boldsymbol{f}(t), \boldsymbol{w})_V - (\mathscr{G}(\boldsymbol{\epsilon}(\boldsymbol{u}_\mu(t)), \zeta_\mu(t)), \boldsymbol{\epsilon}(\boldsymbol{w}))_Q, \quad \forall \boldsymbol{w} \in V. \qquad (11)
\end{aligned}$$

From [9] it follows that problem (11) is equivalent to the following second-kind variational inequality,

$$\begin{aligned}
&(\mathscr{A}(\boldsymbol{\epsilon}(\boldsymbol{v}_\mu(t))), \boldsymbol{\epsilon}(\boldsymbol{w} - \boldsymbol{v}_\mu(t)))_Q + j_\mu(\boldsymbol{w}) - j_\mu(\boldsymbol{v}_\mu(t)) \\
&\quad \geqslant (\boldsymbol{f}(t), \boldsymbol{w} - \boldsymbol{v}_\mu(t))_V - (\mathscr{G}(\boldsymbol{\epsilon}(\boldsymbol{u}_\mu(t)), \zeta_\mu(t)), \boldsymbol{\epsilon}(\boldsymbol{w} - \boldsymbol{v}_\mu(t)))_Q, \quad \forall \boldsymbol{w} \in V, \qquad (12)
\end{aligned}$$

where $j_\mu : V \to \mathbb{R}$ is defined by

$$j_\mu(\boldsymbol{w}) = \int_{\Gamma_C} \mathscr{F}_\mu(w_\tau)\,\mathrm{d}a, \quad \mathscr{F}_\mu(r) = \begin{cases} -gr - g\dfrac{\mu}{2} & \text{if } r < -\mu, \\ \dfrac{g}{2\mu} r^2 & \text{if } r \in [-\mu, \mu], \\ gr - g\dfrac{\mu}{2} & \text{if } r > \mu. \end{cases}$$

Let us consider the following fully discrete problem associated with (12),

$$\begin{aligned}
&(\mathscr{A}(\boldsymbol{\epsilon}(\boldsymbol{v}_\mu^{hk})), \boldsymbol{\epsilon}(\boldsymbol{w}^h - \boldsymbol{v}_\mu^{hk}))_Q + j_\mu(\boldsymbol{w}^h) - j_\mu(\boldsymbol{v}_\mu^{hk})) \\
&\quad \geqslant (\boldsymbol{f}_n, \boldsymbol{w}^h - \boldsymbol{v}_\mu^{hk}))_V - (\mathscr{G}(\boldsymbol{\epsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\epsilon}(\boldsymbol{w}^h - \boldsymbol{v}_\mu^{hk}))_Q, \quad \forall \boldsymbol{w}^h \in V^h, \qquad (13)
\end{aligned}$$

where the subscript $_n$ has been removed in order to simplify the writing. We notice that (13) has a unique solution $\boldsymbol{v}_\mu^{hk} \in V^h$ (see [9]), which can be obtained using the penalty-duality algorithm applied to solve (9).

From [9] we previously knew that $\lim_{\mu \to 0} \|\boldsymbol{v}_n^{hk} - \boldsymbol{v}_\mu^{hk}\|_V = 0$ and, under the assumptions of Corollary 3, after some algebra it implies that

$$\|\boldsymbol{v}_n^{hk} - \boldsymbol{v}_\mu^{hk}\|_V^2 \leqslant c\mu[\|\boldsymbol{v}_n^{hk}\|_V + \|\boldsymbol{v}_\mu^{hk}\|_V] \leqslant c\mu(h + k + \|\boldsymbol{u}\|_{\mathscr{C}([0,T];V)}).$$

5. Numerical simulations

In the examples, the elasticity operator has the form

$$\mathscr{G}(\boldsymbol{\epsilon}(\boldsymbol{u}), \beta) = \beta \Phi(\mathscr{B}\boldsymbol{\epsilon}(\boldsymbol{u})),$$

with Φ defined by $(\Phi(\boldsymbol{\tau}))_{ij} = L$ if $\tau_{ij} > L$, $(\Phi(\boldsymbol{\tau}))_{ij} = \tau_{ij}$ if $\tau_{ij} \in [-L, L]$ and $(\Phi(\boldsymbol{\tau}))_{ij} = -L$ if $\tau_{ij} < -L$, and $\mathscr{B}$ being the two-dimensional elastic stress tensor under plane stress hypothesis:

$$(\mathscr{B}\boldsymbol{\tau})_{\alpha\beta} = \frac{Er}{1 - r^2}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1 + r}\tau_{\alpha\beta}, \quad 1 \leqslant \alpha, \beta \leqslant 2, \quad \boldsymbol{\tau} \in S_2,$$

Table 1
Example **1**: Numerical errors for some n and k

$n \downarrow k \rightarrow$	10.02	10.01	10.005	10.002	10.001
4	19.847	17.495	17.725	17.887	17.943
8	0.92418	0.84569	0.83874	0.83476	0.83346
16	0.25349	0.25351	0.25084	0.24926	0.24874
32	0.13378	0.10731	0.10546	0.10437	0.10401
64	0.12870	0.022878	0.021554	0.020832	0.020607

where E and r represent the Young's modulus and the Poisson's ratio of the material, respectively, and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. Moreover, $\mathscr{A}$ has a similar form (η_1, η_2 are viscosity coefficients),

$$(\mathscr{A}(\boldsymbol{\tau}))_{\alpha\beta} = \eta_1(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \eta_2\tau_{\alpha\beta}, \quad 1 \leqslant \alpha, \beta \leqslant 2, \quad \forall \boldsymbol{\tau} \in S_2.$$

The damage source function used here has the form

$$\phi(\boldsymbol{\epsilon}(\boldsymbol{u}), \zeta) = \lambda_d \left(\frac{1-\zeta}{\eta^*(\zeta)}\right) - \frac{1}{2}\lambda_u \Psi_{q^*}(\boldsymbol{\epsilon}(\boldsymbol{u})) + \lambda_w, \tag{14}$$

with

$$\Psi_{q^*}(\boldsymbol{\tau}) = \begin{cases} |\boldsymbol{\tau}|^2 & \text{if } |\boldsymbol{\tau}|^2 \leqslant q^*, \\ q^* & \text{otherwise,} \end{cases} \qquad \eta^*(s) = \begin{cases} 1 & \text{if } 1 \leqslant s, \\ s & \text{if } \zeta_* \leqslant s \leqslant 1, \\ \zeta_* & \text{if } s \leqslant \zeta_*, \end{cases}$$

being truncation functions introduced for ϕ in order to satisfy the Lipschitz property. A truncation values of $q^* = 1000$, $\zeta_* = 0.01$ as lower limit for the damage and a penalty parameter $\mu = 10^{-9}$ have been used.

5.1. First example

We consider the domain $\Omega = (0, 4) \times (0, 4)$ as the cross-section of a three-dimensional viscoelastic body. On the part $\Gamma_D = \{0\} \times [0, 4]$ the body is rigidly attached and so the displacement field vanishes there, and on $\Gamma_C = (0, 4) \times \{0\}$ the body is in bilateral frictional contact with a rigid obstacle. Horizontal tractions act on the part $\{4\} \times [0, 4] \subset \Gamma_N$ and the part $[0, 4] \times \{4\}$ is traction free. No body forces are assumed to act on the body during the process.

In order to see the convergence behaviour of the scheme, a sequence of numerical solutions is computed based on uniform partitions of the time interval and uniform triangulations of the square $[0, 4] \times [0, 4]$, where $[0, 4]$ is divided into n equal parts. For computations the following data were used:

$$T = 1\,\mathrm{s}, \quad \boldsymbol{f}_B = \boldsymbol{0}\,\mathrm{N/m^3}, \quad \boldsymbol{f}_N(\boldsymbol{x}, t) = (100, 0)e^t\,\mathrm{N/m^2}, \quad g = 5\,\mathrm{N/m^2},$$
$$E = 5000\,\mathrm{N/m^2}, \quad r = 0.2, \quad \eta_1 = 57.69\,\mathrm{N\,s/m^2}, \quad \eta_2 = 38.46\,\mathrm{N\,s/m^2},$$
$$\lambda_D = 0.1, \quad \lambda_u = 1000, \quad \lambda_w = 0, \quad \boldsymbol{u}_0 = \boldsymbol{0}\,\mathrm{m}, \quad \zeta_0(\boldsymbol{x}) = 1 \quad \forall \boldsymbol{x} \in \Omega.$$

The numerical solution corresponding to $n = 128$ and $k = 0.001$ is taken as the "exact" solution used to compute the numerical errors. In Table 1 these errors, obtained for some n and k, are shown.

5.2. Second example

As a second example, a physical setting, similar to that of the above test, has been considered during a time interval of 5 s (i.e., $T = 5$). The boundary Γ is now defined by $\Gamma_D = \emptyset$, $\Gamma_N = \{0, 4\} \times (0, 4)$ and

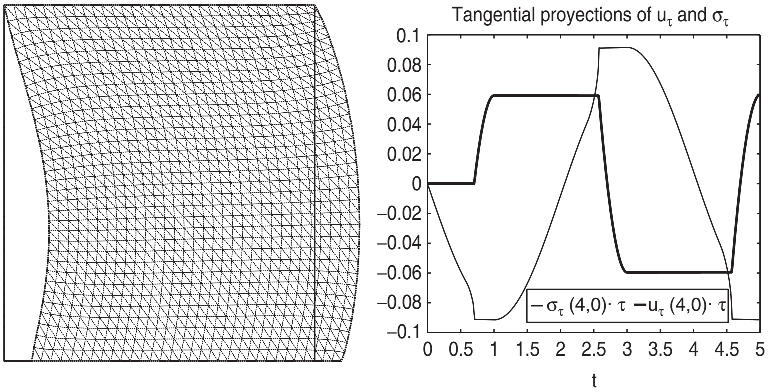

Fig. 2. Example **2**: Deformed configuration (multiplied by 3) at time $t = 5$ s and horizontal displacements at $\boldsymbol{x} = (4, 0)$.

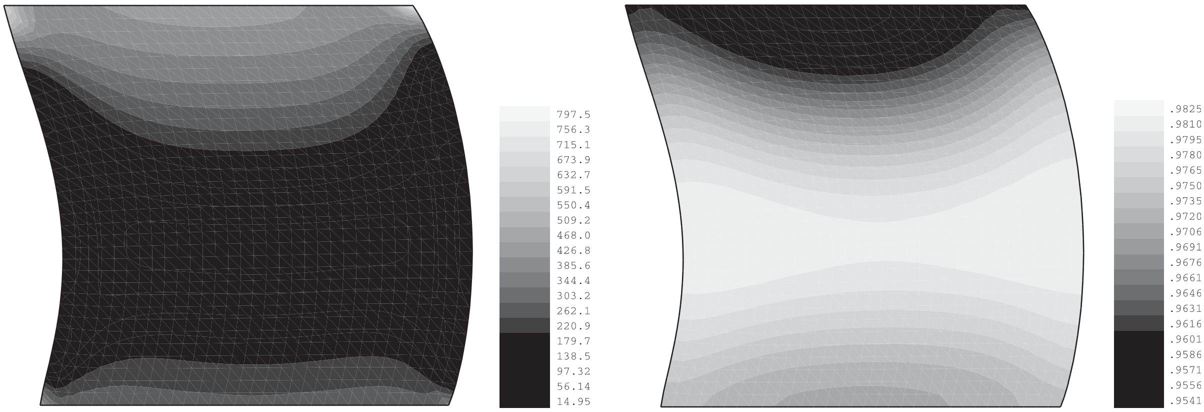

Fig. 3. Example **2**: von Mises stress norm and damage field.

$\Gamma_C = [0, 4] \times \{0, 4\}$, and the body is acted upon by volume forces whose direction oscillates periodically ($\boldsymbol{f}_B(x, y, t) = (100, 0)\sin(\pi t/2)\,\mathrm{N/m^3}$), and it is traction-free on Γ_N.

The friction bound g is now given by $g(x, y) = g_1$ if $y = 0$ and $g(x, y) = g_2$ if $y = 4$, where g_1 and g_2 are friction coefficients for the contact boundary on $[0, 4] \times \{0\}$ and $[0, 4] \times \{4\}$, respectively. In this example, values $g_1 = 45\,\mathrm{N/m^2}$ and $g_2 = 1000\,\mathrm{N/m^2}$ were employed.

The deformed mesh at final time and the initial configuration are shown in Fig. 2 (left-hand side). We notice the movement of the lower horizontal boundary, while the upper one remains clamped (because the tangential stresses do not reach the friction bound on the upper boundary). On the right-hand side, the respective tangential projections of the displacements and shear stresses (divided by an adequate factor) are plotted at point $\boldsymbol{x} = (4, 0)$. It is possible to observe the absence of movement until the friction bound is reached.

Finally, in Fig. 3, the von Mises stress norm and the damage field at final time $t = 5$ are plotted on the deformed configuration. The maximum stresses are located on the upper boundary due to the clamping condition, while on the lower one stresses appear due to the friction. As we expected, the damage is concentrated on the most stressed areas, the contact boundaries.

6. Conclusions

In this paper, a quasistatic viscoelastic contact problem was studied. The contact was modelled using the classical Tresca's friction law. According to [7,8], the effect of the damage was included into the model. The variational formulation led to a coupled system of two nonlinear parabolic variational inequalities. Following [4,10], an existence and uniqueness theorem was stated, based on fixed point arguments and some results on parabolic variational inequalities. Then, a fully discrete scheme, namely Problem $\mathbf{VP}^{hk}$, was introduced using the finite element method and the Euler scheme for approximating the spatial variable and the time derivatives, respectively. Error estimates were provided according to [4,10], from which, under suitable regularity assumptions, the linear convergence of the scheme was derived. The main contribution of this paper concerned the numerical resolution of Problem $\mathbf{VP}^{hk}$, where a penalty of the frictional term was used. Then, a penalty-duality algorithm, introduced in [3], was employed for solving the penalized problem. Finally, two numerical examples were performed to show the accuracy of the algorithm. First, a simple test was considered, dividing an square domain into $2n^2$ triangles. The convergence of the algorithm, depending on the discretization parameters, was clearly observed (see Table 1), although we notice that the linear convergence stated in Corollary 3 was not obtained. Secondly, we considered a test where the contact boundary was divided into two parts with different friction bounds. In Figs. 2 and 3, the results obtained were shown, coinciding with those we previously expected.

References

[1] A. Amassad, M. Shillor, M. Sofonea, A quasistatic contact problem for an elastic perfectly plastic body with Tresca's friction, Nonlinear Anal. 35 (1999) 95–109.
[2] M. Barboteu, W. Han, M. Sofonea, Numerical analysis of a bilateral frictional contact problem for linearly elastic materials, IMA J. Numer. Anal. 22 (3) (2002) 407–436.
[3] A. Bermúdez, C. Moreno, Duality methods for solving variational inequalities, Comput. Math. Appl. 7 (1981) 43–58.
[4] O. Chau, J.R. Fernández, W. Han, M. Sofonea, A frictionless contact problem for elastic-viscoplastic materials with normal compliance and damage, Comput. Methods Appl. Mech. Engrg. 191 (2002) 5007–5026.
[5] P.G. Ciarlet, The finite element method for elliptic problems, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, vol. II, Part 1, North-Holland, Amsterdam, 1991, pp. 17–352.
[6] G. Duvaut, J.L. Lions, Inequalities in Mechanics and Physics, Springer, Berlin, 1976.
[7] M. Frémond, Non-smooth Thermomechanics, Springer, Berlin, 2002.
[8] M. Frémond, B. Nedjar, Damage, gradient of damage and principle of virtual work, Internat. J. Solids and Structures 33 (8) (1996) 1083–1103.
[9] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer, New York, 1984.
[10] W. Han, M. Shillor, M. Sofonea, Variational and numerical analysis of a quasistatic viscoelastic problem with normal compliance, friction and damage, J. Comput. Appl. Math. 137 (2001) 377–398.
[11] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, American Mathematical Society-Intl. Press, Providence, RI, 2002.
[12] R. Liebe, P. Steinmann, A. Benallal, Theoretical and numerical aspects of a thermodynamically consistent framework for geometrically linear gradient damage, Comput. Methods Appl. Mech. Engrg. 190 (2001) 6555–6576.
[13] B. Nedjar, A theoretical and computational setting for a geometrically nonlinear gradient damage modelling framework, Comput. Mech. 30 (2002) 65–80.
[14] P. Steinmann, C. Miehe, E. Stein, Comparison of different finite deformation inelastic damage models within multiplicative elastoplasticity for ductile materials, Comput. Mech. 13 (1994) 458–474.